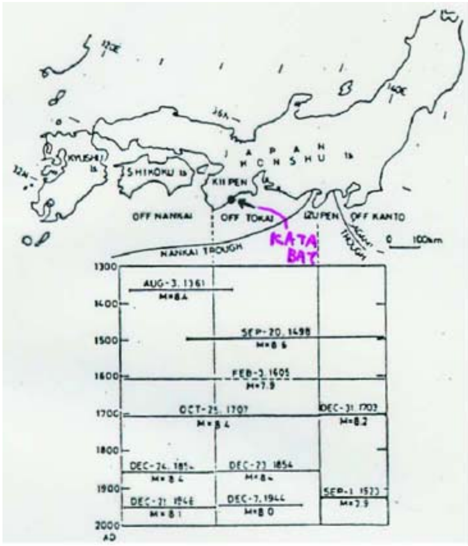
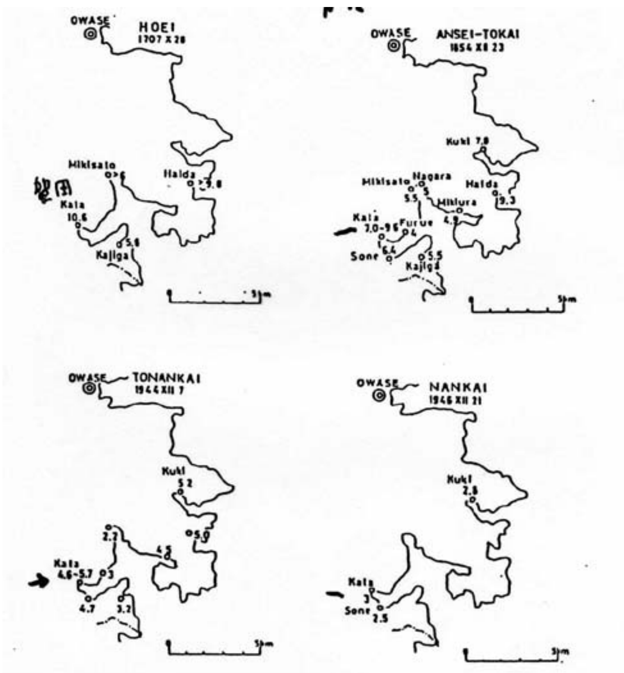
7 . Proper Oscillation in a Bay

0. Introduction

Kata bay is located on the southeast coast of the Kii peninsula, and tsunamis have hit the bay four times in the modern era—in 1707 (Hoei Earthquake), 1854 (Ansei-Tokai Earthquake), 1944 (Tonankai Earthquake), and 1946 (Nankai Earthquake). Kata bay has three sub-divisions, and there are ten villages along its coast. The village Kata is situated on the coast of the western branch; the maximum tsunami height was recorded in this branch every single time.





1. Method of Solving Eigenvalue Oscillation in a Bay

1.1 Formulation of Governing Equation of Water Surface Displacement

In the present chapter, we discuss proper oscillations (eigenvalue oscillation, Seiche) in a bay or in a lake of arbitrary shape and an uneven bottom; moreover, we discuss how a numerical solution can be obtained.

We take a co-ordinate system that covers the bay area, as shown in Fig. 1, and assume an x, y -axis; Then, we set a grid mesh covering the entire bay. One mesh square is counted as a combination of i and j in the x and y directions, respectively.

The equations of motion are given by

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} \qquad (1)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \qquad (2)$$

where (u,v) is the horizontal particle velocity, g is the acceleration due to gravity, and ζ is the displacement of the sea surface.

The equation of mass conservation takes the following form:

$$\frac{\partial \zeta}{\partial t} = -\left\{\frac{\partial (Du)}{\partial x} + \frac{\partial (Dv)}{\partial y}\right\} \qquad (3)$$

Differentiating (3) with t, and substituting u and v by using (1)and (2), (D is a constant for time t) , we have the following equation of motion for ζ:

$$\frac{\partial^2 \zeta}{\partial t^2} = g\left\{\frac{\partial}{\partial x}\left(D\frac{\partial \zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial \zeta}{\partial y}\right)\right\} \qquad (4)$$

We substitute $\zeta = Z(x,y)e^{-i\sigma t}$ and equation (4) becomes

$$-\sigma^2 Z = g\left\{\frac{\partial}{\partial x}\left(D\frac{\partial Z}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial Z}{\partial y}\right)\right\} \qquad (5)$$

We rewrite this differential equation into a difference equation; further, we use $-\frac{l^2\sigma^2}{g} \equiv \lambda$, where l is the grid size. Equation (5) is transferred into the following equation:

$$\lambda Z_{i,j} = \frac{(D_{i+1,j}+D_{i,j})}{2} Z_{i+1,j} + \frac{(D_{i-1}+D_{i,j})}{2} Z_{i-1,j} + \frac{(D_{i,j+1}+D_{i,j})}{2} Z_{i,j+1} + \frac{(D_{i,j-1}+D_{i,j})}{2} Z_{i,j-1}$$

$$-\left(2D_{i,j} + \frac{D_{i+1,j}+D_{i-1,j}+D_{i,j+1}+D_{i,j-1}}{2}\right) Z_{i,j}$$

(6)

Note: Such a transfer of equation (5) is not unique .It is possible to transfer the term of $\frac{\partial}{\partial x}\left(D\frac{\partial Z}{\partial x}\right)$ in different ways. In the present study, we re-write it as follows:

$$\frac{\partial}{\partial x}\left(D\frac{\partial Z}{\partial x}\right) = \frac{\partial D}{\partial x}\frac{\partial Z}{\partial x} + D\frac{\partial^2 Z}{\partial x^2} \quad (6')$$

This is transferred in the following manner:

$$(7) = \frac{\{(D_{i+1}-D_i)\times(Z_{i+1}-Z_i)+(D_i-D_{i-1})\times(Z_i-Z_{i-1})\}}{2l^2} + D_i\frac{(Z_{i+1}-2Z_i+Z_{i-1})}{l^2} \quad (6\text{-a})$$

Equation (6) is derived through this formulation.

On the other hand, it is also possible to transfer in the another manner:

$$(7) = \frac{(D_{i+1}-D_{i-1})}{2l}\times\frac{(Z_{i+1}-Z_{i-1})}{2l} + D_i\frac{(Z_{i+1}-2Z_i+Z_{i-1})}{l^2} \quad (6\text{-b})$$

From this, we obtain an expanded form that is different from (6) as follows:

$$\lambda Z_{i,j} = \frac{D_{i+1,j}+4D_{i,j}-D_{i-1,j}}{4l^2} Z_{i+1,j} + \frac{-D_{i+1,j}+4D_{i,j}+D_{i-1,j}}{4l^2} Z_{i-1,j} +$$

$$+ \frac{D_{i,j+1}+4D_{i,j}-D_{i,j-1}}{4l^2} Z_{i,j+1} + \frac{-4D_{i,j+1}+4D_{i,j}+D_{i,j-1}}{4l^2} - D_{i,j}Z_{i,j}/l^2 \quad (6\text{-x})$$

In the present study, we do not adopt the above form.

1.2 Boundary conditions

We should consider two kinds of boundaries: coastal boundary and open ocean boundary.

(A) Coastal boundary

We assume that we now consider the grid mesh at (i, j), and that the right side of the mesh is a coastline. In this case, since no stream crosses the coastline, we can set $u = 0$ at the coastline. Equation (1) shows that

$\frac{\partial u}{\partial t} = -g\frac{\partial \varsigma}{\partial x}$ Hence, when $u = 0$, we have $\partial\varsigma/\partial x = 0$ at the coastline (a mirror reflection). Thus we have the following coastal boundary condition in the positive x direction.

$$Z_{i+1,j} = Z_{i,j} \tag{7}$$

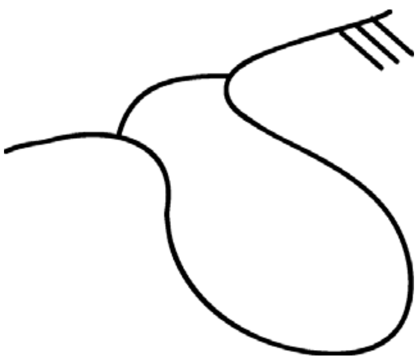
By substituting (7) in (6), we have

$$\lambda Z_{i,j} = \frac{(D_{i-1} + D_{i,j})}{2} Z_{i-1,j} + \frac{(D_{i,j+1} + D_{i,j})}{2} Z_{i,j+1} + \frac{(D_{i,j-1} + D_{i,j})}{2} Z_{i,j-1} - \left(\frac{3}{2} D_{i,j} + \frac{D_{i-1,j} + D_{i,j+1} + D_{i,j-1}}{2}\right) Z_{i,j}$$

This is the expression for the coastline condition on the right side in the form of a finite difference equation.

(B) Open ocean condition

We assume that the upper side of the grid mesh (i, j) is adjacent to the open ocean, where ς is sufficiently small, and that depth in the open ocean area is sufficiently large. This condition can be expressed by using ζ=0, and the depth beyond the boundary has a sufficiently large value. Hence, we use $Z_{i,j+1} = 0$ ($(i, j+1)$ is the mesh location of the upper adjoining grid of the grid (i, j)) and $D_{i,j+1} = 1000 D_{i,j}$.



1.3 One-dimensional Numbering

We re-number the grids of a water area sequentially from the upper row to the bottom and from left to right; the counter is set as k. Thus, the left-most sea grid in the uppermost row has the number $k = 1$, and the adjacent grid on the right has $k = 2$.

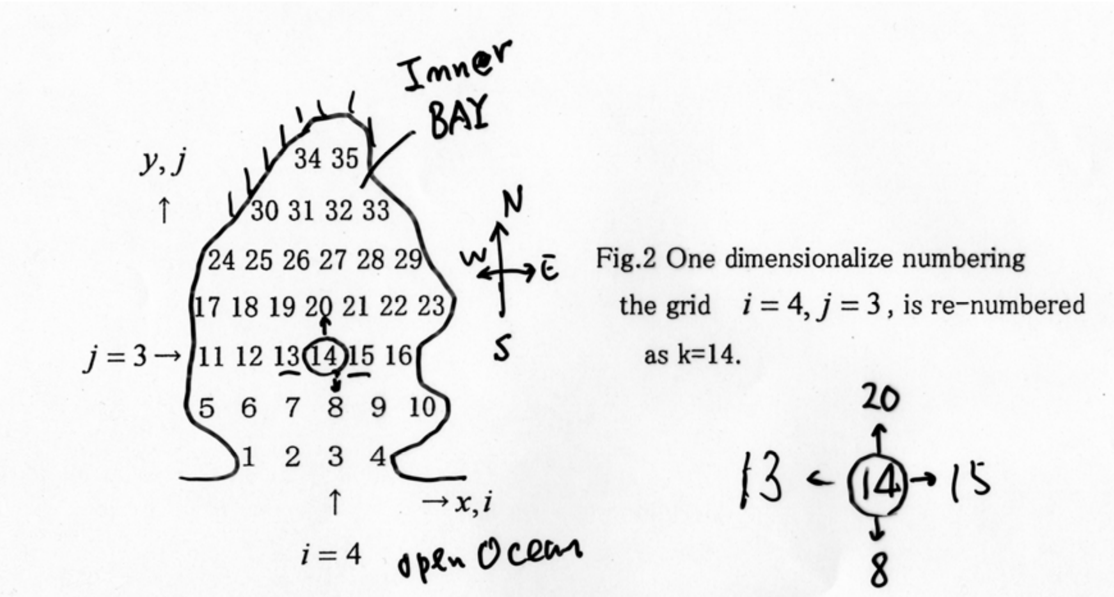


Fig.2 One dimensionalize numbering the grid $i=4, j=3$, is re-numbered as k=14.

Hereafter, we do not use grid (i, j) in equation (6), but instead use k. For example, we can write (6) for the grid of $k=14$ in the following form:

$$\lambda Z_{14} = \frac{D_{15}+D_{14}}{2} Z_{15} + \frac{D_{13}+D_{14}}{2} Z_{13} + \frac{D_8+D_{14}}{2} Z_8 + \frac{D_{20}+D_{14}}{2} Z_{20} - $$

$$- \left\{ 2D_{14} + \frac{D_{13}+D_{15}+D_8+D_{20}}{2} \right\} Z_{14} \qquad (8)$$

Here, we introduce

$(D_{15}+D_{14})/2 = R_{14}$, $(D_8+D_{14})/2 = L_{14}$,

$\frac{(D_{18}+D_{14})}{2} = H_{14}$, $\frac{(D_{20}+D_{14})}{2} = U_{14}$ and $-\left\{ 2D_{14} + \frac{D_{13}+D_{15}+D_8+D_{20}}{2} \right\} = T_{14}$

(Note: $R_{14}+L_{14}+H_{14}+D_{14} = T_{14}$)

1.4 Introduction of Matrix Form

We can express (8) in the following form:

$$(H_{14}, L_{14}, T_{14}, R_{14}, U_{14}) \begin{pmatrix} Z_8 \\ Z_{13} \\ Z_{14} \\ Z_{15} \\ Z_{20} \end{pmatrix} = \lambda Z_{14} \qquad (9)$$

Equation (19) is valid not only for $k=14$, but also for all grids from $k=1$ to $k=37$ (in the case of Fig. 2); thus, (9) can be written in the following form:

$$
\begin{pmatrix}
T_1, R_1, 0,0,...U_{1,} 0,0,0,0,0..........0 \\
L_2, T_2, R_2, 0,0,...,U_2, 0,0,0,........0 \\
0, L_3, T_3, R_3, 0,0,....,U_3, 0,0,.......0 \\
0,0, L_4, T_4, R_4, 0,0,.....,U_4, 0,0,...0 \\
....0 \\
H.......L, T, R,....U.................0 \\
0, H,.......L, T, R,...U,.............0 \\
0,0, H_k,......L_k, T_k, R_k,....U_k,.....0 \\
.... \\
.... \\
0,0,...................., H_{37},.....L_{37}, T_{37}
\end{pmatrix}
\begin{pmatrix}
Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ .... \\ .... \\ .... \\ Z_k \\ .... \\ .... \\ Z_{37}
\end{pmatrix}
= \lambda
\begin{pmatrix}
Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ .... \\ .... \\ .... \\ Z_k \\ .... \\ .... \\ Z_{37}
\end{pmatrix}
\quad (10)
$$

This is a diagonally symmetric matrix with a size of 37 × 37. Tri-diagonal components have non-zero values, and in addition, two components are non-zero in each row (why?). (H_1, L_1 , and R_{37}, U_{37} do not exist. Why?) .

We simply re-write the square matrix in the left side of (10) as A and the one-dimensional vector $\{Z_k\}$ as Z; (10) can then simply be expressed as

$$AZ = \lambda Z \quad (11)$$

(11) is a normal form of the eigenvalue equation of a matrix.

1.5 Banded Storage Matrix

Only five non-zero values appear in one row, and they are arranged within the length of a $2N_b + 1$ $(N_b = j_{\max})$ grid whose central grid is a component of the diagonal elements. Moreover, the matrix is a diagonally symmetric one. Such a matrix is called "a banded symmetric matrix" and 2Nb+1 is the "band width." The size of the original Matrix A is $N \times N$. We can save the necessary memory by using a "banded storage matrix" that has the size of $N \times (N_b + 1)$.

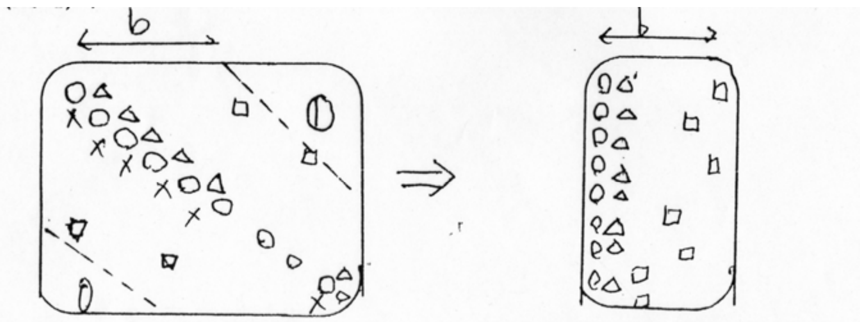


Fig. 3-a Original Matrix Fig. 3-b Banded Storage Matrix

The Fortran subroutine for obtaining the eigenvalue $\lambda_l (l = 1, 2, \cdots, N)$ and the corresponding eigenvector $\{Z_{\lambda,k}\}$ of a given banded symmetric matrix A or its equivalent banded storage matrix A_b is generally present in Fortran's library, and you need not prepare this subroutine on your own.

1.6 Solution of the Eigenvalue Oscillation in a Bay

You can easily select a set of eigenvalues λ_k （$k =$ 1,..., 37）and the corresponding eigenvalue vectors $\{Z_{\lambda,k}\}$. In generally, it is sufficient to select the eigenvalue upto the fifth largest one. Higher-order eigenvalues exist mathematically, but most of the higher-order eigenvalue oscillations are not actually observed in a real sea.

Each eigenvalue $\{\lambda_k\}$ should be negative value, because we defined the λ-value as $-\dfrac{l^2\sigma^2}{g} \equiv \lambda$.

1.7 Oscillation Pattern and Oscillation Period

If we re-arrange a one-dimensional eigenvalue vector into two dimensions $Z_{\lambda,k} \rightarrow Z_\lambda(i, j)$ and then plot it on the map of the bay, we obtain a chart showing the pattern of the eigenvalue oscillation.

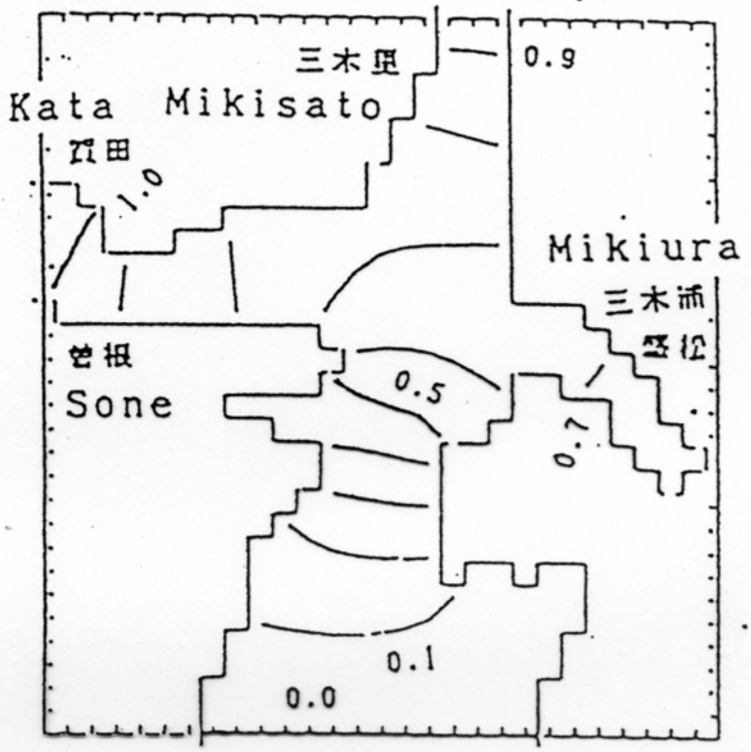
The period of the corresponding oscillations are obtained by

$$\sigma^2 = -\frac{g\lambda}{l^2} \qquad (12)$$

or

$$T = 2\pi\sqrt{-\frac{l^2}{g\lambda}} \qquad (13)$$

MODE I　　T = 21.52



MODE II　　T = 10.32